Bessel function

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In mathematics, **Bessel functions**, first defined by the mathematician Daniel Bernoulli and generalized by Friedrich Bessel, are canonical solutions $y(x)$ of Bessel's differential equation:

$$x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} + (x^2 - \alpha^2)y = 0$$

for an arbitrary real or complex number α. The most common and important special case is where α is an integer, *n*. Then α is referred to as the *order*.

Although α and −α produce the same differential equation, it is conventional to define different Bessel functions for these two orders (e.g., so that the Bessel functions are mostly smooth functions of α).

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Applications

Bessel's equation arises when finding separable solutions to Laplace's equation and the Helmholtz equation in cylindrical or spherical coordinates, and Bessel functions are therefore especially important for many problems of wave propagation, static potentials, and so on. (For cylindrical problems, one obtains Bessel functions of integer order α = *n*; for spherical problems, one obtains half integer orders α = *n*+½.) For example:

- electromagnetic waves in a cylindrical waveguide
- heat conduction in a cylindrical object.
- modes of vibration of a thin circular (or annular) artificial membrane.
- diffusion problems on a lattice.

Bessel functions also have useful properties for other problems, such as signal processing (e.g., see FM synthesis, Kaiser window, or Bessel filter).

Definitions

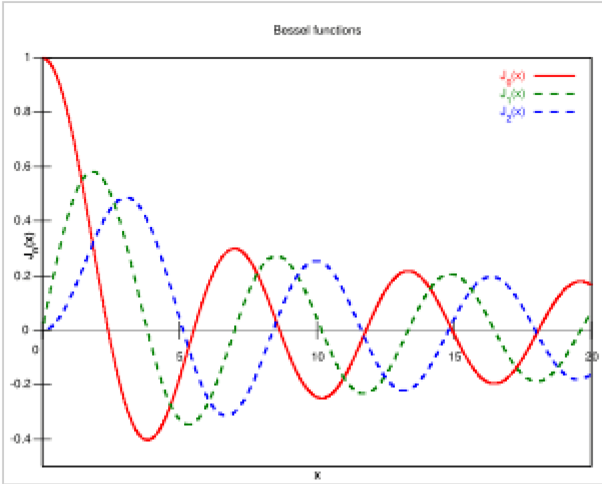
Since this is a second-order differential equation, there must be two linearly independent solutions. Depending upon the circumstances, however, various formulations of these solutions are convenient, and the different variations are described below.

Bessel functions of the first kind

Bessel functions of the first kind, denoted with $J_\alpha(x)$, are solutions of Bessel's differential equation which are finite at $x = 0$ for α an integer or α non-negative. The specific choice and normalization of J_α are defined by its properties below; another possibility is to define it by its Taylor series expansion around $x = 0$ (or a more general power series for non-integer α):

$$J_\alpha(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,\Gamma(m+\alpha+1)} \left(\frac{x}{2}\right)^{2m+\alpha}$$

Here, $\Gamma(z)$ is the gamma function, a generalization of the factorial to non-integer values. The graphs of Bessel functions look roughly like oscillating sine or cosine functions that decay proportionally to $1/\sqrt{x}$ (see also their asymptotic forms, below), although their roots are not generally periodic except asymptotically for large x.



Plot of Bessel function of the first kind, $J_\alpha(x)$, for integer orders α=0,1,2.

If α is not an integer, the functions $J_\alpha(x)$ and $J_{-\alpha}(x)$ are linearly independent and are therefore the two solutions of the differential equation. On the other hand, if the order α is an integer, then the following relationship is valid:

$$J_{-\alpha}(x) = (-1)^\alpha J_\alpha(x)$$

This means that they are no longer linearly independent. The second linearly independent solution is then found to be the Bessel function of the second kind, as discussed below.

Bessel's integrals

Another definition of the Bessel function, for integer values of α, is possible using an integral equation:

$$J_\alpha(x) = \frac{1}{2\pi} \int_0^{2\pi} \cos(\alpha\tau - x\sin\tau)\,d\tau.$$

(For the full expression for real values of α, see Abramowitz and Stegun (1972) page 360)

This is the approach that Bessel used, and from this definition he derived several properties of the function. Another integral representation is:

$$J_\alpha(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-i(\alpha\tau - x\sin\tau)}\,d\tau$$

Relation to hypergeometric series

The Bessel functions can be expressed in terms of the hypergeometric series as

$$J_\alpha(z) = \frac{(z/2)^\alpha}{\Gamma(\alpha+1)}\;{}_0F_1(\alpha+1; -z^2/4).$$

This expression is related to the development of Bessel functions in terms of the Bessel-Clifford function.

Bessel functions of the second kind

The Bessel functions of the second kind, denoted by $Y_\alpha(x)$, are solutions of the Bessel differential equation. They are singular (infinite) at $x = 0$.

$Y_\alpha(x)$ is sometimes also called the **Neumann function**, and is occasionally denoted instead by $N_\alpha(x)$. It is related to $J_\alpha(x)$ by:

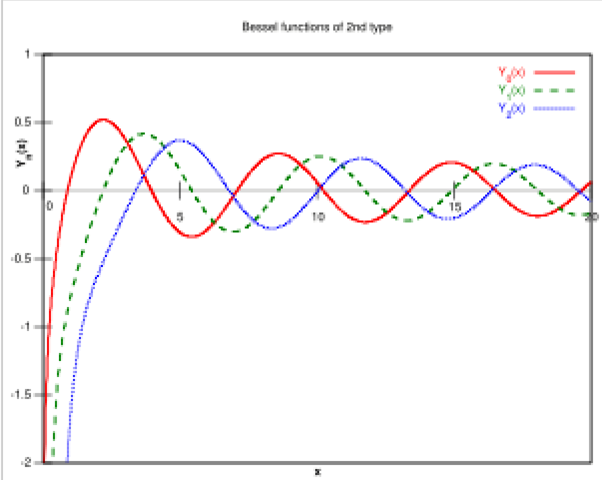
$$Y_\alpha(x) = \frac{J_\alpha(x)\cos(\alpha\pi) - J_{-\alpha}(x)}{\sin(\alpha\pi)},$$

where the case of integer α is handled by taking the limit.

When α is not an integer, the definition of Y_α is redundant (as is clear from its definition above). On the other hand, when α is an integer, Y_α is the second linearly independent solution of Bessel's equation; moreover, as was similarly the case for the functions of the first kind, the following relationship is valid:

$$Y_{-n}(x) = (-1)^n Y_n(x)$$

Both $J_\alpha(x)$ and $Y_\alpha(x)$ are holomorphic functions of x on the complex plane cut along the negative real axis. When α is an integer, there is no branch point, and the Bessel functions are entire functions of x. If x is held fixed, then the Bessel functions are entire functions of α.



Plot of Bessel function of the second kind, $Y_\alpha(x)$, for integer orders α=0,1,2.

Hankel functions

Another important formulation of the two linearly independent solutions to Bessel's equation are the **Hankel functions** $H_\alpha^{(1)}(x)$ and $H_\alpha^{(2)}(x)$, defined by:

$$H_\alpha^{(1)}(x) = J_\alpha(x) + iY_\alpha(x)$$

$$H_\alpha^{(2)}(x) = J_\alpha(x) - iY_\alpha(x)$$

where i is the imaginary unit. These linear combinations are also known as Bessel functions of the third kind; they are two linearly independent solutions of Bessel's differential equation. The Hankel functions of the first and second kind are used to express outward- and inward-propagating cylindrical wave solutions of the cylindrical wave equation, respectively (or vice versa, depending on the sign convention for the frequency). They are named for Hermann Hankel.

Using the previous relationships they can be expressed as:

$$H_\alpha^{(1)}(x) = \frac{J_{-\alpha}(x) - e^{-\alpha\pi i} J_\alpha(x)}{i\sin(\alpha\pi)}$$

$$H_\alpha^{(2)}(x) = \frac{J_{-\alpha}(x) - e^{\alpha\pi i} J_\alpha(x)}{-i\sin(\alpha\pi)}$$

if α is an integer, the limit has to be calculated. The following relationships are valid, whether α is an integer or not:

$$H_{-\alpha}^{(1)}(x) = e^{\alpha\pi i} H_\alpha^{(1)}(x)$$

$$H_{-\alpha}^{(2)}(x) = e^{-\alpha\pi i} H_\alpha^{(2)}(x)$$

Modified Bessel functions

The Bessel functions are valid even for complex arguments x, and an important special case is that of a purely imaginary argument. In this case, the solutions to the Bessel equation are called the **modified Bessel functions** (or occasionally the **hyperbolic Bessel functions**) of the first and second kind, and are defined by:

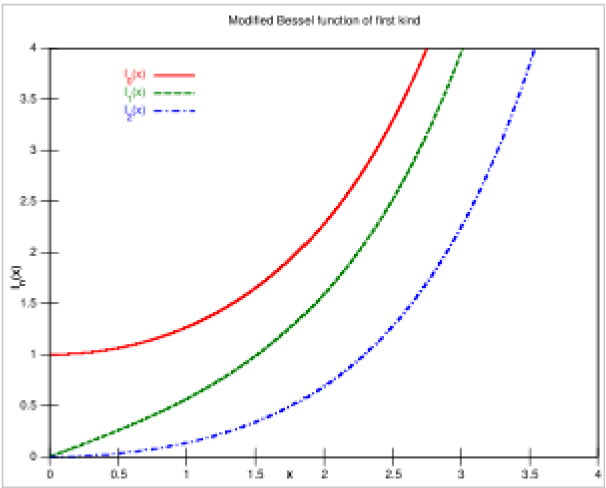
$$I_\alpha(x) = i^{-\alpha} J_\alpha(ix)$$

$$K_\alpha(x) = \frac{\pi}{2} \frac{I_{-\alpha}(x) - I_\alpha(x)}{\sin(\alpha\pi)} = \frac{\pi}{2} i^{\alpha+1} H_\alpha^{(1)}(ix)$$

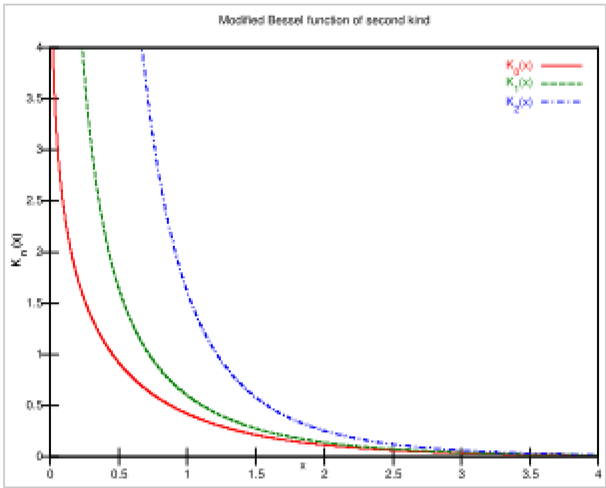
These are chosen to be real-valued for imaginary arguments x. They are the two linearly independent solutions to the modified Bessel's equation:

$$x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} - (x^2 + \alpha^2)y = 0.$$

Unlike the ordinary Bessel functions, which are oscillating as functions of a real argument, I_α and K_α are exponentially growing and decaying functions, respectively. Like the ordinary Bessel function J_α, the function I_α goes to zero at x=0 for $\alpha > 0$ and is finite at x=0 for α=0. Analogously, K_α diverges at x=0.



Modified Bessel functions of 1st kind



Modified Bessel functions of 2nd kind

The **modified Bessel function of the second kind** has also been called by the now-rare names:

- Basset function
- modified Bessel function of the third kind
- MacDonald function

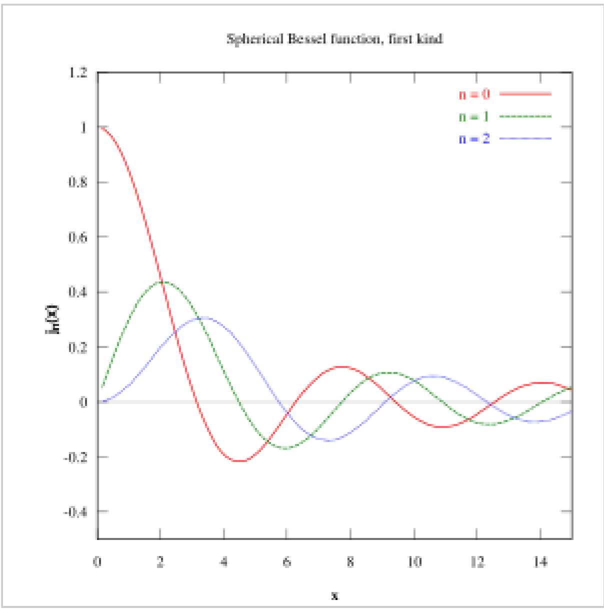
Spherical Bessel functions

When solving the Helmholtz equation in spherical coordinates by separation of variables, the radial equation has the form:

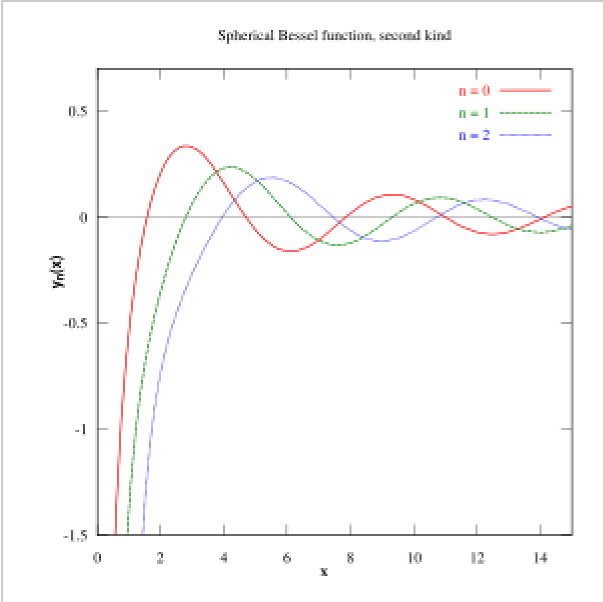
$$x^2 \frac{d^2 y}{dx^2} + 2x \frac{dy}{dx} + [x^2 - n(n+1)]y = 0.$$

The two linearly independent solutions to this equation are called the **spherical Bessel functions** j_n and y_n, and are related to the ordinary Bessel functions J_n and Y_n by:

$$j_n(x) = \sqrt{\frac{\pi}{2x}} J_{n+1/2}(x),$$



Spherical Bessel functions of 1st kind



Spherical Bessel functions of 2nd kind

$$y_n(x) = \sqrt{\frac{\pi}{2x}} Y_{n+1/2}(x) = (-1)^{n+1} \sqrt{\frac{\pi}{2x}} J_{-n-1/2}(x).$$

y_n is also denoted n_n; some authors call these functions the **spherical Neumann functions**.

The spherical Bessel functions can also be written as:

$$j_n(x) = (-x)^n \left(\frac{1}{x}\frac{d}{dx}\right)^n \frac{\sin x}{x},$$
$$y_n(x) = -(-x)^n \left(\frac{1}{x}\frac{d}{dx}\right)^n \frac{\cos x}{x}.$$

The first spherical Bessel function $j_0(x)$ is also known as the (unnormalized) sinc function. The first few spherical Bessel functions are:

$$j_0(x) = \frac{\sin x}{x}$$
$$j_1(x) = \frac{\sin x}{x^2} - \frac{\cos x}{x}$$
$$j_2(x) = \left(\frac{3}{x^2} - 1\right)\frac{\sin x}{x} - \frac{3\cos x}{x^2}$$

and

$$y_0(x) = -j_{-1}(x) = -\frac{\cos x}{x}$$
$$y_1(x) = j_{-2}(x) = -\frac{\cos x}{x^2} - \frac{\sin x}{x}$$
$$y_2(x) = -j_{-3}(x) = \left(-\frac{3}{x^2} + 1\right)\frac{\cos x}{x} - \frac{3\sin x}{x^2}.$$

There are also spherical analogues of the Hankel functions:

$$h_n^{(1)}(x) = j_n(x) + iy_n(x)$$

$$h_n^{(2)}(x) = j_n(x) - iy_n(x).$$

In fact, there are simple closed-form expressions for the Bessel functions of half-integer order in terms of the standard trigonometric functions, and therefore for the spherical Bessel functions. In particular, for non-negative integers *n*:

$$h_n^{(1)}(x) = (-i)^{n+1} \frac{e^{ix}}{x} \sum_{m=0}^{n} \frac{i^m}{m!(2x)^m} \frac{(n+m)!!}{(n-m)!!}$$

and $h_n^{(2)}$ is the complex-conjugate of this (for real *x*). (!! is the double factorial.) It follows, for example, that $j_0(x) = \sin(x)/x$ and $y_0(x) = -\cos(x)/x$, and so on.

Riccati-Bessel functions

Riccati-Bessel functions only slightly differ from spherical Bessel functions:

$$S_n(x) = xj_n(x) = \sqrt{\pi x/2} J_{n+1/2}(x)$$

$$C_n(x) = -xy_n(x) = -\sqrt{\pi x/2} Y_{n+1/2}(x)$$

$$\zeta_n(x) = xh_n^{(2)}(x) = \sqrt{\pi x/2} H_{n+1/2}^{(2)}(x) = S_n(x) + iC_n(x)$$

They satisfy the differential equation:

$$x^2 \frac{d^2y}{dx^2} + [x^2 - n(n+1)]y = 0$$

This differential equation, and the Riccati-Bessel solutions, arises in the problem of scattering of electromagnetic waves by a sphere, known as Mie scattering after the first published solution by Mie (1908). See e.g. Du (2004) for recent developments and references.

Following Debye (1909), the notation ψ_n, χ_n is sometimes used instead of S_n, C_n.

Asymptotic forms

The Bessel functions have the following asymptotic forms for non-negative α. For small arguments $0 < x \ll \sqrt{\alpha + 1}$, one obtains:

$$J_\alpha(x) \rightarrow \frac{1}{\Gamma(\alpha+1)} \left(\frac{x}{2}\right)^\alpha$$

$$Y_\alpha(x) \rightarrow \begin{cases} \frac{2}{\pi}\left[\ln(x/2) + \gamma\right] & \text{if } \alpha = 0 \\ -\frac{\Gamma(\alpha)}{\pi}\left(\frac{2}{x}\right)^\alpha & \text{if } \alpha > 0 \end{cases}$$

where γ is the Euler-Mascheroni constant (0.5772...) and Γ denotes the gamma function. For large arguments $x \gg |\alpha^2 - 1/4|$, they become:

$$J_\alpha(x) \rightarrow \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{\alpha\pi}{2} - \frac{\pi}{4}\right)$$

$$Y_\alpha(x) \rightarrow \sqrt{\frac{2}{\pi x}} \sin\left(x - \frac{\alpha\pi}{2} - \frac{\pi}{4}\right).$$

(For α=1/2 these formulas are exact; see the spherical Bessel functions above.) Asymptotic forms for the other types of Bessel function follow straightforwardly from the above relations. For example, for large $x \gg |\alpha^2 - 1/4|$, the modified Bessel functions become:

$$I_\alpha(x) \rightarrow \frac{1}{\sqrt{2\pi x}} e^x,$$

$$K_\alpha(x) \rightarrow \sqrt{\frac{\pi}{2x}} e^{-x}.$$

while for small arguments $0 < x \ll \sqrt{\alpha + 1}$, they become:

$$I_\alpha(x) \rightarrow \frac{1}{\Gamma(\alpha+1)} \left(\frac{x}{2}\right)^\alpha$$

$$K_\alpha(x) \rightarrow \begin{cases} -\ln(x/2) - \gamma & \text{if } \alpha = 0 \\ \frac{\Gamma(\alpha)}{2} \left(\frac{2}{x}\right)^\alpha & \text{if } \alpha > 0 \end{cases}$$

Properties

For integer order α = *n*, J_n is often defined via a Laurent series for a generating function:

$$e^{(x/2)(t-1/t)} = \sum_{n=-\infty}^\infty J_n(x) t^n,$$

an approach used by P. A. Hansen in 1843. (This can be generalized to non-integer order by contour integration or other methods.) Another important relation for integer orders is the **Jacobi-Anger identity**:

$$e^{iz\cos\phi} = \sum_{n=-\infty}^\infty i^n J_n(z) e^{in\phi},$$

which is used to expand a plane wave as a sum of cylindrical waves, or to find the Fourier series of a tone modulated FM signal.

The functions J_α, Y_α, $H_\alpha^{(1)}$, and $H_\alpha^{(2)}$ all satisfy the recurrence relations:

$$Z_{\alpha-1}(x) + Z_{\alpha+1}(x) = \frac{2\alpha}{x} Z_\alpha(x)$$

$$Z_{\alpha-1}(x) - Z_{\alpha+1}(x) = 2\frac{dZ_\alpha}{dx}$$

where *Z* denotes *J*, *Y*, $H^{(1)}$, or $H^{(2)}$. (These two identities are often combined, e.g. added or subtracted, to yield various other relations.) In this way, for example, one can compute Bessel functions of higher orders (or higher derivatives) given the values at lower orders (or lower derivatives). In particular, it follows that:

$$\left(\frac{d}{xdx}\right)^m [x^\alpha Z_\alpha(x)] = x^{\alpha-m} Z_{\alpha-m}(x)$$

$$\left(\frac{d}{xdx}\right)^m \left[\frac{Z_\alpha(x)}{x^\alpha}\right] = (-1)^m \frac{Z_{\alpha+m}(x)}{x^{\alpha+m}}$$

Modified Bessel functions follow similar relations :

$$e^{(x/2)(t+1/t)} = \sum_{n=-\infty}^{\infty} I_n(x) t^n,$$

and

$$e^{z\cos\theta} = I_0(z) + 2\sum_{n=1}^{\infty} I_n(z)\cos(n\theta),$$

The recurrence relation reads

$$C_{\alpha-1}(x) - C_{\alpha+1}(x) = \frac{2\alpha}{x} C_\alpha(x)$$

$$C_{\alpha-1}(x) + C_{\alpha+1}(x) = 2\frac{dC_\alpha}{dx}$$

where C denotes I or K. These recurrence relations are useful for discrete diffusion problems.

Because Bessel's equation becomes Hermitian (self-adjoint) if it is divided by x, the solutions must satisfy an orthogonality relationship for appropriate boundary conditions. In particular, it follows that:

$$\int_0^1 x J_\alpha(x u_{\alpha,m}) J_\alpha(x u_{\alpha,n}) dx = \frac{\delta_{m,n}}{2} [J_{\alpha+1}(u_{\alpha,m})]^2 = \frac{\delta_{m,n}}{2} [J'_\alpha(u_{\alpha,m})]^2,$$

where $\alpha > -1$, $\delta_{m,n}$ is the Kronecker delta, and $u_{\alpha,m}$ is the m-th zero of $J_\alpha(x)$. This orthogonality relation can then be used to extract the coefficients in the Fourier-Bessel series, where a function is expanded in the basis of the functions $J_\alpha(x\, u_{\alpha,m})$ for fixed α and varying m. (An analogous relationship for the spherical Bessel functions follows immediately.)

Another orthogonality relation is the *closure equation*:

$$\int_0^\infty x J_\alpha(ux) J_\alpha(vx) dx = \frac{1}{u}\delta(u-v)$$

for $\alpha > -1/2$ and where δ is the Dirac delta function. For the spherical Bessel functions the orthogonality relation is:

$$\int_0^\infty x^2 j_\alpha(ux) j_\alpha(vx) dx = \frac{\pi}{2u^2}\delta(u-v)$$

for $\alpha > 0$.

Another important property of Bessel's equations, which follows from Abel's identity, involves the Wronskian of the solutions:

$$A_\alpha(x)\frac{dB_\alpha}{dx} - \frac{dA_\alpha}{dx}B_\alpha(x) = \frac{C_\alpha}{x},$$

where A_α and B_α are any two solutions of Bessel's equation, and C_α is a constant independent of x (which depends on α and on the particular Bessel functions considered). For example, if $A_\alpha = J_\alpha$ and $B_\alpha = Y_\alpha$, then C_α is $2/\pi$. This also holds for the modified Bessel functions; for example, if $A_\alpha = I_\alpha$ and $B_\alpha = K_\alpha$, then C_α is -1.

(There are a large number of other known integrals and identities that are not reproduced here, but which can be found in the references.)

See also

- Bessel-Clifford function

References

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External Links

- Using the Bessel function, scientists write text on water surface. [1] (http://www.pinktentacle.com/2006/07/device-uses-waves-to-print-on-water-surface/)

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